

The Hilbert-Smith Conjecture

by

Louis F. McAuley

Dedicated to the memory of Deane Montgomery

Abstract

The Hilbert-Smith Conjecture states that if G is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then G is a Lie group. A rather straightforward proof of this conjecture is given. The motivation is work of Cernavskii (“Finite-to-one mappings of manifolds”, *Trans. of Math. Sk.* 65 (107), 1964.) His work is generalized to the orbit map of an effective action of a p -adic group on compact connected n -manifolds with the aid of some new ideas. There is no attempt to use Smith Theory even though there may be similarities.

It is well known that if a locally compact group acts effectively on a connected n -manifold M and G is not a Lie group, then there is a subgroup H of G isomorphic to a p -adic group A_p which acts effectively on M . It can be shown that A_p can not act effectively on an n -manifold and, hence, The Hilbert Smith Conjecture is true. The existence of a non empty fixed point set adds some complexity to the proof. In this paper, it is shown that A_p can not act freely on a compact connected n -manifold. The basic ideas for the general case are more clearly seen in this case. The general proof will be given in another paper.

1. Introduction.

In 1900, Hilbert proposed twenty-three problems [8]. For an excellent discussion concerning these problems, see the *Proceedings of Symposia In Pure Mathematics* concerning “Mathematical Developments Arising From Hilbert Problems” [3]. The abstract by C.T.

Yang [22] gives a review of Hilbert's Fifth Problem "*How is Lie's concept of continuous groups of transformations of manifolds approachable in our investigation without the assumption of differentiability?*" Work of von Neumann [40] in 1933 showed that differentiability is not completely dispensable. This with results of Pontryagin [35] in 1939 suggested the specialized version of Hilbert's problem: *If G is a topological group and a topological manifold, then is G topologically isomorphic to a Lie group?* This is generally regarded as Hilbert's Fifth Problem. The first partial result was given by Brouwer [26] in 1909-1910 for locally euclidean groups of dimension ≤ 2 . The best known partial results were given for compact locally euclidean groups and for commutative locally euclidean groups by von Neumann [40] and Pontryagin [35], respectively.

In 1952, the work of Gleason [5] and Montgomery-Zippin [13] proved: *Every locally euclidean group is a Lie group.* This solved Hilbert's Fifth Problem.

A more general version of Hilbert's Fifth Problem is the following:

If G is a locally compact group which acts effectively on a connected manifold as a topological transformation group, then is G a Lie group? The Hilbert-Smith Conjecture states that the answer is yes.

Papers of Montgomery [34] in 1945 and Bochner-Montgomery [1] in 1946 established the partial result: *Let G be a locally compact group which acts effectively on a differentiable manifold M such that for any $g \in G$, $x \mapsto gx$ is a differentiable transformation of M . Then G is a Lie group and (G, M) is a differentiable transformation group.* Another partial result was given by a theorem of Yamabe [43] and a Theorem of Newman [15] as follows: *If G is a compact group which acts effectively on a manifold and every element of G is of finite order, then G is a finite group.*

It has been shown [14] that an affirmative answer to the generalized version of Hilbert's Fifth Problem is equivalent to a negative answer to the following: *Does there exist an effective action of a p -adic group on a manifold?*

It can be proved that the answer to this question is No. However, in this paper it is shown that there is *no free action of a p -adic group on a connected compact n -manifold*. The existence of a non empty fixed point set complicates the argument even though the techniques used are the same as in the free case. The proof is more easily understood in the free case. The general case is not as difficult to follow once the free case is understood. This will follow in another paper.

A brief review of some of the consequences of efforts to solve this problem is given below. There are examples in the literature of *effective* actions of an infinite compact 0-dimensional topological group G (*each $g \in G - \{\text{identity}\}$ moves some point*) on locally connected continua. The classic example of Kolmogoroff [29] in 1937, is one where G operates effectively but not *strongly effectively* [24] on a 1-dimensional locally connected continuum (Peano continuum) such that the orbit space is 2-dimensional. In 1957, R.D. Anderson [24] proved that *any* compact 0-dimensional topological group G can act strongly effectively as a transformation group on the (Menger) universal 1-dimensional curve M such that either (1) the orbit space is homeomorphic to M or (2) the orbit space is homeomorphic to a regular curve.

In 1960, C.T. Yang [44] proved that if a p -adic group, A_p , acts effectively as a transformation group on X (a locally compact Hausdorff space of homology dimension not greater than n), then the homology dimension of the orbit space X/A_p is not greater than $n + 3$. If X is an n -manifold, then the homology dimension of X/A_p is $n + 2$. If A_p acts strongly effectively (freely) on an n -manifold X , then the dimension of X/A_p is either $n + 2$ or infinity. At about the same time (1961), Bredon, Raymond, and Williams [25] proved the same results using different methods. There are, of course, actions by p -adic groups on p -adic solenoids and actions by p -adic solenoids on certain spaces. See [25] for some of these results.

In 1961, Frank Raymond published the results of his study of the orbit space M/A_p *assuming* an effective action by A_p (as a transformation group) on an n -manifold M .

Later (1967), Raymond [38] published work on two problems in the theory of generalized manifolds which are related to the (generalized) Hilbert Fifth Problem.

In 1963, Raymond and Williams [39] gave examples of compact metric spaces X^n of dimension n and an action by a p -adic group, A_p , on X^n such that $\dim X^n/A_p = n + 2$. Work related to and used in [39] is the paper [41] by Williams. In [41], Williams answers a question of Anderson [24; p. 799] by giving a free action by a compact 0-dimensional group G on a 1-dimensional Peano continuum P with $\dim P/G = 2$.

In 1976, I described [32; 33] what I called p -adic polyhedra which admit periodic homeomorphisms of period p . Proper inverse systems $\{P_i, \phi_i\}$ of p -adic n_i -polyhedra have the property that the inverse limit $X = \varprojlim P_i$ admits a free action by a p -adic group.

In 1980, one of my students, Alan J. Coppola [28] generalized results of C.T. Yang [44] which involve homologically analyzing p -adic actions. Coppola formalized these so that homological calculations could be done in a more algorithmic manner. He defined a p -adic transfer homomorphism and used it to produce all of the relevant Smith-Yang exact sequences which are used to homologically analyze Z_{p^r} -actions on compact metric spaces. Coppola studied p -adic actions on homologically uncomplicated spaces. In particular, he proved that if X is a compact metric A_p -space of homological dimension no greater than n and X is homologically locally connected, then the $(n + 3)$ -homology of any closed subset $A \subset X/A_p$ vanishes.

In 1983, Robinson and I proved Newman's Theorem for finite-to-one open and closed mappings on manifolds [10]. We formalized Newman's Property (and variations) and studied this property for discrete open and closed mappings on generalized continua in 1984 [11].

In 1985, H-T Ku, M-C Ku, and Larry Mann investigated in [30] the connections between Newman's Theorem involving the size of orbits of group actions on manifolds and

the Hilbert-Smith Conjecture. They establish Newman's Theorem (Newman's Property [11]) for actions of compact *connected* non-Lie groups such as the p -adic solenoid.

In 1997, D. Repovš and E.V. Ščepin [51] gave a proof of the Hilbert-Smith Conjecture for actions by Lipschitz maps. See also related work by Shchepin [52]. In the same year, Iozke Maleshick [53] proved the Hilbert-Smith conjecture for Hölder actions.

In 1999, Gaven J. Martin [54] announced a proof of The Hilbert-Smith Conjecture for quasiconformal actions on Riemannian manifolds and related spaces.

The crucial idea that works here is M.H.A. Newman's idea used in his proof that for a given compact connected n -manifold M , there is an $\epsilon > 0$ such that if h is any periodic homeomorphism of period p , a prime > 1 , of M onto itself, then there is some $x \in M$ such that the orbit of x , $\{x, h(x), \dots, h^{p-1}(x)\}$, has diameter $\geq \epsilon$. It is well known that the collection of orbits under the action of a transformation group G on a compact Hausdorff space X is a *continuous decomposition* of X .

The works [20; 21] of David Wilson and John Walsh [18] show that there exist continuous decompositions of n -manifolds M^n , $n \geq 3$, into Cantor sets. This paper shows that such decompositions can not be equivalent to those induced by any action of a p -adic transformation group A_p acting on M^n .

I owe a special debt of gratitude to Patricia Tulley McAuley who has been extremely helpful in reading drafts of numerous attempts to solve this problem and who has provided helpful insights with regard to Čech homology. Also, I am indebted to the work of Cernavskii [27].

OUTLINE OF A PROOF

It is well know that if a locally compact group G acts effectively on a connected n -manifold M and G is not a Lie group, then there is a subgroup H of G isomorphic to a p -adic group A_p which acts effectively on M . Thus, the Hilbert-Smith Conjecture can be established by proving that there is no effective action by a p -adic group A_p on a

connected n -manifold M .

It is proved that if $L(M, p)$, p is a prime greater than 1, is the class of all orbit mappings $\phi : M \rightarrow M/A_p$ where A_p acts freely on a compact connected n -manifold M , then M has Newman's Property w.r.t. $L(M, p)$. That is, there is $\epsilon > 0$ such that for any $\phi \in L(M, p)$, there is some $x \in M$ such that $\text{diam } \phi^{-1}\phi(x) \geq \epsilon$ (using the metric on M). This yields a contradiction to the well known fact that if $L(M, p) \neq \emptyset$, then for given $\epsilon > 0$, there is $\phi \in L(M, p)$ such that $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for any $x \in M$.

Lemma 2. (A consequence of a Theorem of Floyd [4].) Suppose that M is a compact connected n -manifold. There is a finite open covering W_1 of M such that (1) order $W_1 = n + 1$ and (2) there is a finite open refinement W_2 of W_1 which covers M such that if W is any finite open covering of M refining W_2 , then $\pi_{W_1} : \check{H}_n(M) \rightarrow H_n(W_1)$ maps $\check{H}_n(M)$ isomorphically onto the image of the projection $\pi_{WW_1} : H_n(W) \rightarrow H_n(W_1)$.

[Here, if U is either a finite or locally finite open covering of M , then $H_n(U)$ is the n^{th} simplicial homology group of the nerve $N(U)$ of U . The coefficient group is always Z_p and $\check{H}_n(M)$ denotes the n^{th} Čech homology of M .]

Now, choose $U = W_1$ and a finite open covering W of M which star refines W_2 where W_1 and W_2 satisfy Lemma 2.

Let ϵ be the Lebesgue number of W_2 . Choose $\phi \in L(M, p)$ such that $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Construct the special coverings $\{V^m\}$ and the special refinements $\{U^m\}$ as in Lemma 4 below where V^1 star refines W_2 such that order $U^m = n + 1$ and with projections α_m , β_m , and π_m yielding the following commutative diagram:

$$\begin{array}{ccccccc}
\leftarrow & H_n(V^m) & \xleftarrow{\beta_m^*} & H_n(U^m) & \xleftarrow{\alpha_m^*} & H_n(V^{m+1}) & \xleftarrow{\beta_{m+1}^*} & H_n(U^{m+1}) & \leftarrow \\
& \nu_m \uparrow & \beta_m^* \swarrow & & \nwarrow \alpha_m^* & \nu_{m+1} \uparrow & \beta_{m+1}^* \swarrow & & \nwarrow \alpha_{m+1}^* \\
\leftarrow & H_n(V_n^m) & \xleftarrow{\pi_m^*} & & & H_n(V_n^{m+1}) & \xleftarrow{\pi_{m+1}^*} & & H_n(V_n^{m+2}) \leftarrow
\end{array}$$

Here ν_m is the natural map of an n -cycle in $H_n(V_n^m)$, the n^{th} simplicial homology group of the n -skeleton of the nerve $N(V^m)$ of V^m , into its homology class in $H_n(V^m)$. The other

maps are those induced by the projections α_m , β_m and π_m . The upper sequence, of course, yields the Čech homology group $\check{H}_n(M)$ as its inverse limit. Furthermore, it can be easily shown, using the diagram, that $\check{H}_n(M)$ is isomorphic to the inverse limit $G = \varprojlim H_n(V_n^m)$, of the lower sequence. Specifically, $\gamma : \check{H}_n(M) \rightarrow G$ defined by $\gamma(\Delta) = \{\beta_m^*(\pi_{U^m}(\Delta))\}$ is an isomorphism of $\check{H}_n(M)$ onto G . We shall use the isomorphism in what follows and for convenience we shall let $\gamma(\Delta) = \{z_m^n(\Delta)\}$, i.e. $z_m^n(\Delta) = \beta_m^*(\pi_{U^m}(\Delta)) \in H_n(V_n^m)$. The group $\varprojlim H_n(V_n^m)$ is used because the operator σ (introduced later) is applied to actual n -cycles rather than to elements of a homology class. This is the reason for the sequence $\{U^m\}$. Note that there is no attempt to use Smith Theory.

An operator σ_m is defined on the n chains of $N(V^{m+1})$ for each m . The operator σ_m maps n -cycles to n -cycles and commutes with the projections $\pi_m^* : H_n(V_n^{m+1}) \rightarrow H_n(V_n^m)$ and, hence, induces an automorphism on $\pi_{V_n^{m+1}}(\check{H}_n(M)) \subset H_n(V_n^{m+1})$. See Lemmas 5 and 6.

In the general case, it is shown in Lemma 10 that $\text{int } F_\phi = \emptyset$ where $F_\phi = \{x \mid \phi^{-1}\phi(x) = x\}$. Here, $F_\phi = \emptyset$.

Distinguished families of n -simplices in $N(V^m)$ are defined. Now, let $z_m = z_m^n(\Delta)$ where Δ is the generator of $G \cong Z_p$. For each n -simplex δ^n in $z_m(\Delta)$, there is a unique distinguished family S_j^m of n -simplices in $N(V^m)$ which contains δ^n . If C_j is the collection of all n -simplices in $z_m(\Delta)$ which are in S_j^m , then the sum of the coefficients of those members of C_j (as they appear in $z_m(\Delta)$) is 0 mod p . Take the projection $\pi_{V^m U}$ from V^m to $U = W_1$. Hence, $\pi_{V^m U}$ has the property that all members of a distinguished family S_j^m of n -simplices in $N(V^m)$ project to the same simplex in $N(W_1)$. Thus, the projection of those members of $z_m(\Delta)$ which are in S_j^m project to the same simplex δ_j in $N(W_1)$ and the coefficient of δ_j is 0 mod p . Thus, $\pi_{V^m U} : H_n(V_n^m) \rightarrow H_n(U)$ takes the nontrivial n -cycle $z_m(\Delta)$ to the 0- n cycle mod p . This violates the conclusion of Lemma 2. Thus, M has Newman's Property w.r.t. the class $L(M, p)$. Hence, ϵ is a Newman's number.

It is well known that if A_p acts effectively (freely, in this case) on a compact connected n -manifold M , then given any $\epsilon > 0$, there is a free action of A_p on M such that $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. That is, M fails to have Newman's property w.r.t. the class $L(M, p)$. It follows that A_p can not act freely on a compact connected n -manifold M .

Details of the proof follow.

2. *Some properties of the orbit mapping of an effective action by A_p on a compact connected orientable n -manifold M .*

Suppose that ϕ is the orbit mapping of a p -adic group A_p acting as a transformation group on a connected compact n -manifold $M^n = M$ where p is a prime larger than 1. By [12; 21], there is a sequence $A_p = H_0 \supset H_1 \supset H_2 \supset \dots$ of open (and closed) subgroups of A_p which closes down on the identity e of A_p such that when $j > i$, H_i/H_j is a cyclic group of order p^{j-i} . Let $h_{ij} : A_p/H_j \rightarrow A_p/H_i$ and $h_i : A_p \rightarrow A_p/H_i$ be homomorphisms induced by the inclusion homomorphisms (quotient homomorphisms) on A_p and A_p/H_j . Then $\{A_p/H_i; h_{ij}\}$ is an inverse system and $\{h_i\}$ gives an isomorphism of A_p onto $\varprojlim A_p/H_i$. Now, let $a \in A_p - H_i$. For each natural number i , let a_i be the coset aH_i in A_p/H_i . Then a_i is a periodic homeomorphism of M/H_i onto M/H_i with a_i^q being the identity mapping where $q = p^i$ is the period of a_i . Consequently, H_i acts as a transformation group on M and A_p/H_i acts as a cyclic transformation group on M/H_i .

As above, let $\{H_i\}$ be a sequence of open (and closed) subgroups of A_p such that (a) $H_i \supset H_{i+1}$ for each i , (b) if $j \geq i$, then H_i/H_j is a cyclic group of order p^{j-i} , and (c) A_p/H_i is a cyclic group of order p^i . Since A_p acts effectively on M (a compact connected n -manifold), the cyclic group A_p/H_i acts effectively on M/H_i with orbit space M/A_p . Let $\pi_{ij} : M/H_j \rightarrow M/H_i$ where $j > i$ and $\pi_i : M \rightarrow M/H_i$ be maps induced by the identity map of M . Also, $\pi_i = \pi_{ji} \circ \pi_j$. Thus, $\{M/H_i : \pi_{ij}\}$ is an inverse system and $\{\pi_i\}$ gives a homeomorphism of M onto the inverse limit M/H_i ($\varprojlim M/H_i$). Notice that

H_i/H_j , a cyclic group of order p^{j-i} , acts on M/H_j with orbit space M/H_i [cf. 23; p. 211] where $\pi_i = \pi_{ji} \circ \pi_j$ and $\pi_{ji} : M/H_j \rightarrow M/H_i$ is the orbit map of the action of H_i/H_j on M/H_j . Notice that M/H_j is the orbit space of the action of H_j on M where H_j is isomorphic to A_p . It follows that if $\epsilon > 0$, then there is a natural number j such that $H_j \cong A_p$ acts (effectively) on M such that if $\phi_j : M \rightarrow M/H_j$ is the orbit map of the action, then $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Observe that if H is a non trivial open and closed subgroup of A_p , then for some i , $H = H_i$.

The following lemma is crucial to defining certain coverings of M with distinguished families of open sets.

Lemma 1. Suppose that ϕ is the orbit mapping $\phi : M \rightarrow M/A_p$ where A_p acts effectively (freely, in this paper) on a compact connected n -manifold M . For each $z \in M/A_p$ and $\epsilon > 0$, there is a connected set U such that (1) $\text{diam } U < \epsilon$, (2) $z \in U$, and (3) $\phi^{-1}(U) = \{U_1, U_2, \dots, U_{p^s}\}$ where s is a natural number such that (a) U_i is a component of $\phi^{-1}(U)$ for each i , $1 \leq i \leq p^s$, (b) $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, (c) $\phi(U_i) = U$ for each i , and (d) U_1 is homeomorphic to U_j for each j , $1 < j \leq p^s$ (by maps compatible with the projections $\phi|_{U_j}$). [The homeomorphism taking U_1 to U_j is a power of a fixed element $g \in A_p - H_1$ and is used in Lemma 5.]

Proof. Since $z \in M/A_p$, $\phi^{-1}(z)$ is non degenerate. For $\epsilon > 0$, there is a connected open set U such that (1) $\text{diam } U < \epsilon$, (2) $z \in U$, and (3) $\phi^{-1}(U)$ consists of a finite number (larger than one) of components U_1, U_2, \dots, U_m such that $\bar{U}_i \cap \bar{U}_j = \emptyset$ for $i \neq j$, and $\phi(U_i) = U$ for each i . This follows by Whyburn's Theory of open mappings [19, pp. 78-80].

For each U_j , a component of $\phi^{-1}(U)$, there is an open and closed subgroup G_j of A_p which is the largest subgroup of A_p which leaves U_j invariant and the map induced by ϕ maps U_j/G_j onto U . Since G_j is a normal subgroup of A_p , $G_i = G_j$ for each i and j . Furthermore, A_p/G_j is a cyclic group of order p^s where s is a natural number. There are p^s pairwise distinct components of $\phi^{-1}(U)$. (See [36: Lemma 2]). It follows that $G_1 = H_i$ for some i where $\{H_i\}$ is the sequence of open and closed subgroups of A_p

which closes down on the identity $e \in A_p$ (mentioned above) and $s = i$. Let $a \in A_p - H_i$ such that aH_i generates the cyclic group A_p/H_i . For each natural number i , let a_i be the coset aH_i in A_p/H_i . Thus, a_i is a periodic homeomorphism of M/H_i onto M/H_i with $a_i^q = e$ where $q = p^i$ is the period of a_i .

Let $f_i : M \rightarrow M/H_i$ be the orbit map of the action of H_i on M and $g_i : M/H_i \rightarrow M/A_p$ be the orbit map of the action of the cyclic group A_p/H_i on M/H_i . That is, $\phi = g_i f_i$. There are p^i cosets $\{v_m H_i\}_{m=1}^{p^i}$ where $v_1 = e$ (the identity) such that for each $x \in U_1$, $\phi^{-1}\phi(x) = \bigcup_{m=1}^{p^i} v_m H_i(x)$ where $v_m H_i(x) = \{h(x) \mid h \in v_m H_i\}$, (2) $v_m H_i(x) \in U_m/H_i$, (3) v_m is an orientation preserving homeomorphism of M onto M , and (4) if $A_p/H_i = \{a_i, a_i^1, a_i^2, \dots, a_i^{p^i-1}\}$ (a cyclic group), then there are elements k_1, k_2, \dots, k_{p^i} where $k_1 = e$ such that (a) $k_m(\pi_i(x)) = \pi_i(v_m H_i(x))$ where π_i maps $\phi^{-1}(U)$ onto $\phi^{-1}(U)/H_i$ and (b) k_m maps U_1/H_i homeomorphically onto U_m/H_i with $k_m = v_m H_i \in A_p/H_i$ which is a homeomorphism of M/H_i onto M/H_i . Thus, $v_m(x) \in U_m$. Let $z \in U_m$. Hence, $\pi_i(z) \in U_m/H_i$ and $(v_m H_i)^{-1}(\pi_i(z)) = v_m^{-1} H_i(\pi_i(z)) \in U_1/H_i$. Consequently, $v_m^{-1} H_i(\pi_i(z)) = \pi_i(v_m^{-1}(z)) \in U_1/H_i$ which implies that $v_m^{-1}(z) \in U_1$. Finally, $v_m(v_m^{-1}(z)) = z$ and v_m maps U_1 homeomorphically onto U_m .

Lemma 1 is proved.

3. Special coverings and distinguished families.

Let $L(M, p) = \{\phi \mid \phi \text{ is the orbit mapping of a free action of a } p\text{-adic group } A_p \text{ (} p \text{ a prime with } p > 1 \text{) on a compact connected metric } n\text{-manifold without boundary, } \phi : M \rightarrow M/A_p\}$. It would simplify the proof of lemmas which follow to know that M is triangulable. Without this knowledge, a theorem of E.E. Floyd is used.

Notation. Throughout this paper, $\check{H}_n(X)$ will denote the n^{th} Čech homology group of X with coefficients in Z_p , the integers mod p , p a fixed prime larger than 1. Also, $H_n(K)$ will denote the n^{th} simplicial homology of a finite simplicial complex K , with coefficients

in Z_p . If U is a finite open covering of a space X , then $N(U)$ denotes the nerve of U , $H_n(U)$ is the n^{th} simplicial homology group of $N(U)$, and π_U the usual projection homomorphism $\pi_U : \check{H}_n(X) \rightarrow H_n(U)$.

Definition. If f is a mapping of M onto Y , then an open covering U of M is said to be *saturated* (more precisely, saturated w.r.t. f) iff for each $u \in U$, $f^{-1}f(u) = u$. That is, u is an open inverse set.

The next lemma follows.

Lemma 2. Suppose that M is a compact connected metric n -manifold. There is a finite open covering W_1 of M such that (1) order $W_1 = n + 1$ and (2) there is a finite open refinement W_2 of W_1 which covers M such that if W is any finite open covering of M refining W_2 , then $\pi_{W_1} : \check{H}_n(M) \rightarrow H_n(W_1)$ maps $\check{H}_n(M)$ isomorphically onto the image of the projection $\pi_{WW_1} : H_n(W) \rightarrow H_n(W_1)$.

Proof. Adapt Theorem (3.3) of [4] to the situation here and use (2.5) of [4].

If M is triangulable, then there is a sufficiently fine triangulation T such that if U consists of the open stars of the vertices of T , then $\pi_U : \check{H}_n(M) \rightarrow H_n(U)$ is an isomorphism onto (where, of course, $\check{H}_n(M) \cong Z_p$).

NOTE: The reader can assume, for convenience, that M is triangulable and that W_1 is the collection of open stars of a sufficiently fine triangulation T such that the n^{th} simplicial homology, $H_n(W_1)$, with coefficients in Z_p , of the nerve, $N(W_1)$, of W_1 is Z_p .

Standing Hypothesis: In the following, M is a compact and connected metric n -manifold. Also, $L(M, p)$ is as defined above. The finite open coverings W_1 and W_2 which satisfy Lemma 2 will be used in certain lemmas and constructions which follow. Suppose also that $Y = M/A_p$ has a countable basis $Q = \{B_i\}_{i=1}^{\infty}$ such that (a) for each i , B_i is connected and uniformly locally connected and (b) if H is any subcollection of Q and

$\bigcap_{h \in H} h \neq \emptyset$, then $\bigcap_{h \in H} h$ is connected and uniformly locally connected (a consequence of a theorem due to Bing and Floyd [50]).

Lemma 3. Suppose that $\phi \in L(M, p)$. Then there exists a finite open covering R of $Y = M/A_p$ such that (a) if $y \in Y$, then there is $r \in R$ such that $y \in r$, $r \in Q$ where Q is the basis in The Standing Hypothesis, $\phi^{-1}(r) = r_1 \cup r_2 \cup \dots \cup r_q$, $q = p^t$ for some natural number t , such that for each $i = 1, 2, \dots, q$, r_i is a component of $\phi^{-1}(r)$, r_i maps onto r under ϕ , $\overline{r_i} \cap \overline{r_j} = \emptyset$ for $i \neq j$, and r_i is homeomorphic to r_j for each i and j with a homeomorphism compatible with the projection ϕ (indeed, there is an element of A_p which takes r_i onto r_j), (b) R is irreducible, (c) $V = \{c \mid c \text{ is a component of } \phi^{-1}(r) \text{ for some } r \in R\}$ is such that V star refines W_2 , and (d) if $r_x \in R$, $r_y \in R$, $r_x \cap r_y \neq \emptyset$, $\phi^{-1}(r_x)$ consists of exactly p^{m_x} components, $\phi^{-1}(r_y)$ consists of exactly p^{m_y} components, and $m_x \geq m_y$, then each component of $\phi^{-1}(r_y)$ meets exactly $p^{m_x - m_y}$ components of $\phi^{-1}(r_x)$.

Proof. Since Y is compact, use Lemma 1 to obtain a finite irreducible covering R of Y of sets r satisfying the conditions of the lemma such that R star refines $\{\phi(u) \mid u \in W_2\}$. Property (d) of the conclusion of Lemma 3 is satisfied by using the compactness of Y and choosing R such that each $r \in R$ has sufficiently small diameter and $r \in Q$ (the basis in The Standing Hypothesis). The lemma is established.

Lemma 3 is just the first step in establishing Lemma 4 below.

Lemma 4. There are sequences $\{V^m\}$ and $\{U^m\}$ of finite open coverings of M cofinal in the collection of all open coverings of M such that (1) V^{m+1} star refines U^m , (2) V^1 star refines W_2 of Lemma 2, (3) U^m star refines V^m , (4) order $U^m = n + 1$, (5) V^m is generated by a finite open covering R^m of $Y = M/A_p$, (6) V^m and R^m have the properties stated in Lemma 3 where R^m replaces R and V^m replaces V , (7) $\{\text{mesh } V^m\} \rightarrow 0$, and (8) there are projections $\pi_m : V^{m+1} \rightarrow V^m$ such that (a) $\pi_m = \beta_m \alpha_m$ where $\alpha_m : V^{m+1} \rightarrow U^m$ and $\beta_m : U^m \rightarrow V^m$, (b) π_m takes each distinguished family $\{f_{ij}^m\}_{j=1}^{t_i^{m+1}}$ in V^{m+1} (defined in a manner like those defined for V^1 and V^2 below) onto a distinguished family $\{f_{sj}^m\}_{j=1}^{t_s^m}$ in V^m , and (c) π_m extends to a simplicial mapping (also, π_m) of $N(V^{m+1})$ into $N(V^m)$. (Also, α_m and β_m denote the extensions of α_m and β_m

to simplicial mappings $\alpha_m : N(V^{m+1}) \rightarrow N(U^m)$ and $\beta_m : N(U^m) \rightarrow N(V^m)$ where $\pi_m = \beta_m \alpha_m$.)

NOTE: As stated in The Standing Hypothesis, Y has a countable basis $Q = \{B_i\}_{i=1}^\infty$ such that (a) for each i , B_i is connected and uniformly locally connected and (b) if H is any subcollection of Q and $\bigcap_{h \in H} h \neq \emptyset$, then $\bigcap_{h \in H} h$ is connected and uniformly locally connected (a consequence of a theorem due to Bing and Floyd [50]). Each $r \in R^m$ can be chosen from Q .

The proof of Lemma 4, although straightforward, is long and tedious. The existence of $V = V^1$ in Lemma 3 (which star refines W_2) generated by $R = R^1$ is an initial step of a proof using mathematical induction. Additional first steps are described below. These should help make it clear how the induction is completed to obtain a proof of Lemma 4.

Call the collection $\{r_1, r_2, \dots, r_q\}$ consisting of all components of $\phi^{-1}(r)$ a *distinguished* family in V (defined in Lemma 3) generated by $r \in R$ where R satisfies Lemma 3. The finite open covering V can be partitioned into the subcollections $\{r_1, r_2, \dots, r_q\}$ consisting of the components of $\phi^{-1}(r)$ for some $r \in R$ which are defined to be *distinguished* families generated by $r \in R$. Thus, V is generated by $R = R^1$.

Let $V^1 = V$. Clearly, V^1 star refines W_2 . Note that order V^1 may be larger than $n + 1$ since if ϕ is the orbit mapping of an effective action by a p -adic transformation group, then $\dim Y = n + 2$ or ∞ [22].

The covering $V^1 = V$ is defined to be a *special covering* of M w.r.t. ϕ generated by R . Of course, ϕ is fixed throughout this discussion as in the statement of Lemma 4. Observe that it follows from Lemma 1, that if $\{f_{kj}^1\}_{j=1}^q$ and $\{f_{mj}^1\}_{j=1}^s$ are two distinguished families (non degenerate) in V^1 such that for some i and t , $f_{ki}^1 \cap f_{mt}^1 \neq \emptyset$, then for each j , the number of elements of $\{f_{mj}^1\}_{j=1}^s$ which have a non empty intersection with f_{kj}^1 is a constant c_k and for each j , the number of elements of $\{f_{kj}^1\}_{j=1}^q$ which have a non empty intersection with f_{mj}^1 is a constant c_m where $c_k = p^b$, $b \geq 0$, and $c_m = p^d$, $d \geq 0$.

Construction Of U^1 Of Order $n + 1$ Which Refines V^1

The reason that the sequence $\{U^m\}$ is constructed is to prove (using the definitions of α_m , β_m , and $\pi_m = \beta_m \alpha_m$) that the inverse limit of the n^{th} simplicial homology of the n -skeleta of the nerve of V^m is Z_p which permits the application of σ (defined below) to actual n -cycles. The operator σ can not be applied (as defined) to elements of a homology class.

*The next step is to describe a special refinement U^1 of V^1 which has order $n + 1$ and other crucial properties. First, construct U_1^1 . List the distinguished families of V^1 as $F_1^1, F_2^1, \dots, F_{n_1}^1$ where $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ where $t_i^1 = p^{c_i}$ which are non degenerate. Recall that f_{ij}^1 is homeomorphic to f_{it}^1 for each i, j , and t that makes sense. Since R^1 (which generates V^1) is irreducible, it follows that if $f_{ij}^1 \in F_i^1$ and $f_{st}^1 \in F_s^1$ where F_i^1 and F_s^1 are distinguished families in V^1 with $i \neq s$, then f_{ij}^1 and f_{st}^1 are *independent*, that is, $f_{ij}^1 \not\supset f_{st}^1$ and $f_{st}^1 \not\supset f_{ij}^1$. This ordering may be changed below.*

For each i , $1 \leq i \leq n_1$, choose a closed and connected subset K_i in $\phi(f_{i1}^1) = r_i \in R^1$ where $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ such that (1) $K_{ij} = \phi^{-1}(K_i) \cap f_{ij}^1$, K_{ij} is homeomorphic to K_{is} for any s and j that makes sense, (2) $C = \{\text{int } K_{ij} \mid 1 \leq i \leq n_1 \text{ and } 1 \leq j \leq t_i^1\}$ covers M and $K = \{\text{int } K_i \mid 1 \leq i \leq n_1\}$ covers $M/A_p = Y$, and (3) K_{ij} is connected, $1 \leq j \leq t_i^1$. To see that this is possible, choose a closed subset A_i of r_i , $1 \leq i \leq n_1$, such that $A = \{A_i \mid 1 \leq i \leq n_1\}$ covers Y [47; 49]. Note that there exists a natural number k such that $\{A_i = r_i - N_{\frac{1}{k}}(\partial r_i) \mid 1 \leq i \leq n_1\}$ covers Y . To see this, suppose that for each k , there is $x_k \notin \bigcup_{i=1}^{n_1} (r_i - N_{\frac{1}{k}}(\partial r_i))$. Since Y is compact, there is a subsequence $\{x_{n(k)}\}$ of $\{x_k\}$ which converges to $x \in r_q$ for some q . There is some m such that $x \in r_q - N_{\frac{1}{m}}(\partial r_q)$ and $x \in \text{interior}(r_q - N_{\frac{1}{m+1}}(\partial r_q))$ which leads to a contradiction. Choose $p_i \in r_i$ and m sufficiently large such that K_i is the component of $r_i - N_{\frac{1}{m}}(\partial r_i)$ which contains p_i . For each m , let $C_m(p_i)$ be the component of $r_i - N_{\frac{1}{m}}(\partial r_i)$ which contains a fixed $p_i \in r_i$ (with m large enough). It will be shown that $\bigcup_{m=1}^{\infty} C_m(p_i) = r_i$. Suppose that there is $q \in r_i$ such that $q \notin \bigcup_{m=1}^{\infty} C_m(p_i)$. Since r_i is uniformly locally connected and locally compact (\bar{r}_i is a

Peano continuum), there is a simple arc $p_i q$ from p_i to q in r_i . Consequently, for some m , $r_i - N_{\frac{1}{m}}(\partial r_i) \supset p_i q$ which is in $C_m(p_i)$. This is contrary to the assumption above. Hence, $\bigcup_{m=1}^{\infty} C_m(p_i) = r_i$. For $p_i \in r_i$ fixed as above, choose $p_{ij} \in \phi^{-1}(p_i) \cap f_{ij}^1$ for $1 \leq j \leq t_i^1$. Let $C_m^i(p_{ij})$ be the component of $\phi^{-1}(C_m(p_i))$ which is in f_{ij}^1 and contains p_{ij} . It will be shown that $\bigcup_{m=1}^{\infty} C_m^i(p_{ij}) = f_{ij}^1$. If this is false, then there is a $q_j \in f_{ij}^1 - \bigcup_{m=1}^{\infty} C_m^i(p_{ij})$. Since f_{ij}^1 is ulc and locally compact, there is a simple arc $p_{ij} q_j$ from p_{ij} to q_j in f_{ij}^1 . Now, $r_i \supset \phi(p_{ij} q_j)$. For m large enough, $C_m(p_i) \supset \phi(p_{ij} q_j)$ and some component of $\phi^{-1}\phi(p_{ij} q_j)$ contains $p_{ij} q_j$ and lies in $C_m^i(p_{ij})$. This is contrary to the assumption above. Hence, $\bigcup_{m=1}^{\infty} C_m^i(p_{ij}) = f_{ij}^1$. Choose m sufficiently large that condition (2) above is satisfied where $K_i = C_m(p_i)$ and (4) int $C_m(p_i) \supset A_i$. By [19], $\phi^{-1}(C_m(p_i))$ consists of a finite number of components each of which maps onto K_i under ϕ . Choose m sufficiently large that the components of $\phi^{-1}(K_i)$ are $\{K_{ij}\}_{j=1}^{t_i^1}$ and conditions (1) - (4) are satisfied.

By a Theorem (Nagami and Roberts) and its Corollary [49; pp. 90-91], a metric space X has $\dim X \leq n$ if and only if X has a sequence $\{G_i\}$ of open coverings of X such that (1) G_{i+1} refines G_i for each i , (2) order $G_i \leq n+1$ for each i , and (3) mesh $G_i < \frac{1}{i}$. If X is locally connected, then the elements of G_i can be chosen to be connected for each i . If X is an n -manifold, then the elements of G_i can be chosen to be connected for each i . If X is a triangulable n -manifold, then it is easy to see this by using barycentric subdivisions of a triangulation of X . Choose such a sequence $\{G_i\}$ of open coverings of M such that G_1 star refines $K = \{\text{int } K_i \mid 1 \leq i \leq n_1\}$.

The next step is to show how to choose some G_i from which U_1^1 is chosen and later modified to give U^1 with the desired properties.

Let ϵ be the Lebesgue number of the covering K . Choose t such that (1) mesh $G_t < \frac{1}{t}$, (2) if $g \in G_t$, then $\text{diam } \phi(g) < \frac{\epsilon}{8}$, and (3) if $y \in Y$ and $G(y) = \{g \mid g \in G_t \text{ and } y \in \phi(g)\}$, then there is s , $1 \leq s \leq n_1$, such that $\text{int } K_s \supset \bigcup_{g \in G(y)} \overline{\phi(g)}$. (Note that (3) follows from the choice of ϵ , t , and the fact that G_t star refines K .) Choose $U_1^1 \subset G_t$

such that U_1^1 is an irreducible finite covering of M . Since G_t is an open covering of M with connected open sets, mesh $G_t < d$, and order $G_t \leq n + 1$, U_1^1 has the following properties:

- (1) order $U_1^1 \leq n + 1$ and U_1^1 star refines V^1 ,
- (2) if $u \in U_1^1$, then u is connected, and
- (3) if $y \in Y$ and $G(y) = \{g \mid g \in U_1^1 \text{ and } y \in \phi(g)\}$, then there is s , $1 \leq s \leq n_1$, such that $\text{int } K_s \supset \bigcup_{g \in G(y)} \overline{\phi(g)}$.

Construction Of U^1 Which Refines U_1^1 In A Special Way

For each $y \in Y$, let $Q(y) = \{\phi(u) \mid u \in U_1^1 \text{ and } y \in \phi(u)\}$. There are at most a finite number of such sets distinct from each other. Order these sets as Q_1, Q_2, \dots, Q_{m_1} such that for $i < j$, $Q_i \neq Q_j$ and $\text{card } Q_i \geq \text{card } Q_j$. Let $O_i = \cap Q_i - \bigcup_{j < i} (\cap Q_j)$ where $\cap Q_i = \{x \mid x \in \phi(u) \text{ for each } \phi(u) \in Q_i, 1 \leq i \leq m_1\}$. For each i , $1 \leq i \leq m_1$, let $B_{i\phi(u)} = (\partial O_i) \cap \partial \phi(u)$ where $\phi(u) \in Q_i$ and $(\partial O_i) \cap \partial \phi(u) \neq \emptyset$. There are at most a finite number of such non empty sets distinct from each other. Let B_1, B_2, \dots, B_{m_2} denote all those sets distinct from each other. Let $B = \bigcup_{i=1}^{m_2} B_i$. For each $y \in B$, let $D(y) = \{B_t \mid y \in B_t\}$. There are at most a finite number of such non empty sets distinct from each other. Order these as D_1, D_2, \dots, D_{m_3} such that if $i < j$, then $D_i \neq D_j$ and $\text{card } D_i \geq \text{card } D_j$. For each i , $1 \leq i \leq m_2$, there is $u \in U_1^1$ and a closed subset C_{iu} of ∂u such that $\phi(C_{iu}) = B_i$.

It follows from the definition that $\phi^{-1}(B)$ is closed and contains no open set. Hence, dimension $\phi^{-1}(B) \leq n - 1$. For each $y \in Y$, choose $0 < \epsilon_y < (\frac{1}{8}) \min\{\rho(y, B_i) \mid y \notin B_i, 1 \leq i \leq m_2\} \cup \{\rho(\cap D_i, \cap D_j) \mid (\cap D_i) \cap (\cap D_j) = \emptyset\}$ and cover B with a finite irreducible collection E of ϵ_y -neighborhoods for $y \in B$. Choose t sufficiently large such that if $g \in G_t$ and $\phi(g) \cap B \neq \emptyset$, then there is $e \in E$ such that $e \supset \overline{\phi(g)}$. Let H' be a finite irreducible collection of open sets which refines G_t , covers $\phi^{-1}(B)$, refines U_1^1 , and order $H' \leq n$ (see [48; pp. 133-34]). In addition, choose H' such that H' can be extended to a finite

irreducible cover G of M (i.e., $G \supset H'$) such that order $G \leq n+1$, G refines U_1^1 , and if $g \in G - H'$, then $\bar{g} \cap \phi^{-1}(B) = \emptyset$. Now, cover $\phi^{-1}(B)$ with a finite irreducible collection H of open sets such that (a) H refines H' , (b) order $H \leq n$, (c) order $(H \cup H') \leq n+1$ (if $N[\delta^{n-1}]$ is the nucleus of an $(n-1)$ -simplex in the nerve, $N(H')$, of H' , then one and only one element of H contains $N[\delta^{n-1}] \cap \phi^{-1}(B)$), and (d) if $O = \{g \cap \phi^{-1}(O_i) \mid g \in G \text{ and } 1 \leq i \leq m_1\}$ (recall that $\phi^{-1}(B) \cap \phi^{-1}(O_i) = \emptyset$), then $U^1 = H \cup O$ has the following properties: (1) U^1 refines U_1^1 , (2) order $U^1 = n+1$, (3) if $(\cap D_i) \cap (\cap D_j) = \emptyset$, $u \in U^1$, $v \in U^1$, $\phi(u) \cap (\cap D_i) \neq \emptyset$, and $\phi(v) \cap (\cap D_j) \neq \emptyset$, then $\overline{\phi(u)} \cap \overline{\phi(v)} = \emptyset$, (4) if $B_i \cap B_j = \emptyset$, $u \in U^1$, $v \in U^1$, $\phi(u) \cap B_i \neq \emptyset$, and $\phi(v) \cap B_j \neq \emptyset$, then $\overline{\phi(u)} \cap \overline{\phi(v)} = \emptyset$ (a consequence of (3)), (5) if $u \in U^1$, $\phi(u) \cap (\cap D_i) \neq \emptyset$, and $\phi(u) \cap (\cap D_j) \neq \emptyset$, then $(\cap D_i) \cap (\cap D_j) \neq \emptyset$, (6) if $u \in U^1$ and $\phi(u) \cap B = \emptyset$ where $B = \bigcup_{j=1}^{m_2} B_j$, then $O_i \supset \phi(u)$ for some i , $1 \leq i \leq n_1$ (it follows that $\phi(h) \supset \phi(u)$ for all $h \in U_1^1$ such that $\phi(h) \cap \phi(u) \neq \emptyset$), and (7) if $u \in U^1$ and $\phi(u) \cap (\cap D_t) \neq \emptyset$ for the smallest t , then $\phi(h) \supset \phi(u)$ for all $h \in U_1^1$ such that there is no $B_x \in D_t$ with the property that $\partial\phi(h) \supset B_x$ and $\phi(h) \cap \phi(u) \neq \emptyset$. Observe that if $u \in U^1$ and $\phi(u) \cap B = \emptyset$, then for some i , $O_i \supset \phi(u)$. It is not difficult to see that U^1 has properties (1) - (7).

Construction of V^2 Which Refines U^1

For $y \in B \subset Y$ choose $r_y \in Q$ (the basis for Y described above) such that (1) $\bigcap_{u \in Q(y)} \phi(u) \supset \bar{r}_y$, (2) if $U(y) = \{u \mid u \in U^1 \text{ (not } U_1^1) \text{ and } y \in \phi(u)\}$, then $\bigcap_{u \in U(y)} \phi(u) \supset \bar{r}_y$, (3) $\text{diam } r_y < (\frac{1}{8}) \min\{\rho(y, \partial\phi(v)) \mid v \in U_1^1 \text{ and } y \notin \partial\phi(v)\}$, (4) $y \in r_y$, (5) $\phi^{-1}(r_y) = r_{y1} \cup r_{y2} \cup \dots \cup r_{yq}$, $q = p^{t_q}$ where $t_q \geq 1$, r_{yi} maps onto r_y under ϕ , $\bar{r}_{yi} \cap \bar{r}_{yj} = \emptyset$ for $i \neq j$, and r_{yi} is homeomorphic to r_{yj} for each i and j with a homeomorphism compatible with the projection ϕ (indeed, there is an element of A_p which takes r_{yi} onto r_{yj}), and (6) for each i , there is $u \in U^1$ such that $u \supset \bar{r}_{yi}$. See the proof of Lemma 3 for the construction. Let R_1^2 denote a finite irreducible collection of such sets r_y which covers B such that $H \supset U(y)$. If $y \in Y - (R_1^2)^*$, then choose r_y satisfying (1) - (6) above

such that $\bar{r}_y \cap B = \emptyset$. Let R_2^2 denote a finite irreducible collection of such r_y described above which covers $Y - (R_1^2)^*$. Let $R^2 = R_1^2 \cup R_2^2 = \{r_{y_1}, r_{y_2}, \dots, r_{y_{n_2}}\}$ which is a finite irreducible covering of Y . Now, $V^2 = \{c \mid c \text{ is a component of } \phi^{-1}(r_{y_i}) \text{ for some } i, 1 \leq i \leq n_2\}$. Also, V^2 can be chosen such that V^2 star refines U^1 . The collection of components, $\{f_{ij}^2\}_{j=1}^{t_i^2}$, of $\phi^{-1}(r_{y_i})$ is defined to be a distinguished family in V^2 .

Definitions Of U^1 , α_1 , β_1 , And $\pi_1 = \beta_1 \alpha_1$

Let $F_i^2 = \{f_{ij}^2\}_{j=1}^{t_i^2}$ be any distinguished family (nondegenerate) in V^2 , $1 \leq i \leq n_2$.

By definition, r_{y_i} in R^2 generates F_i^2 .

Case (1): $y_i \in B = \bigcup_{u \in U_1^1} \partial\phi(u)$ where $r_{y_i} \in R_1^2$. Let $c_i = \min\{t \mid y_i \in \cap D_t\}$. Let $s_i = \min\{s \mid \text{int } K_s \supset \overline{\phi(u)} \text{ for all } u \in U_1^1 \text{ such that } u \in G(y_i)\}$. Choose $F_{s_i}^1 = \{f_{s_{ij}}^1\}_{j=1}^{t_{s_i}^1}$ for F_i^2 . For each j , $1 \leq j \leq t_i^2$, choose $U_{ij} \in H$ such that $U_{ij} \supset \bar{f}_{ij}^2$. There is a unique z_{ij} , $1 \leq z_{ij} \leq t_{s_i}^1$, such that $f_{s_{iz_{ij}}}^1 \supset U_{ij} \supset f_{ij}^2$. Let $\alpha_1(f_{ij}^2) = U_{ij}$, $\beta_1(U_{ij}) = f_{s_{iz_{ij}}}^1$, and $\pi_1 = \beta_1 \alpha_1$.

Case (2): $y_i \notin B$. Let $s_i = \min\{s \mid \text{int } K_s \supset \bigcup_{u \in G(y_i)} \overline{\phi(u)}\}$. Choose $F_{s_i}^1 = \{f_{s_{ij}}^1\}_{j=1}^{t_{s_i}^1}$ for F_i^2 . For each j , $1 \leq j \leq t_i^2$, choose $U_{ij} \in O \subset U^1$ such that $U_{ij} \supset \bar{f}_{ij}^2$. There is a unique z_{ij} , $1 \leq z_{ij} \leq t_{s_i}^1$, such that $f_{s_{iz_{ij}}}^1 \supset U_{ij} \supset f_{ij}^2$. Let $\alpha_1(f_{ij}^2) = U_{ij}$, $\beta_1(U_{ij}) = f_{s_{iz_{ij}}}^1$, and $\pi_1 = \beta_1 \alpha_1$.

It will be shown that the mappings α_1 and β_1 are well defined. If β_1 is not well defined, then there exists F_i^2 and F_k^2 , two different distinguished families in V^2 such that (a) $s_i \neq s_k$, (b) $F_{s_i}^1$ is chosen for F_i^2 , (c) $F_{s_k}^1$ is chosen for F_k^2 , and (d) $U_{ij} = U_{kt} \supset f_{ij}^2 \cup f_{kt}^2$ where U_{ij} is chosen from U^1 such that $U_{ij} \supset \bar{f}_{ij}^2$ and $U_{kt} = U_{ij}$ is chosen such that $U_{kt} \supset \bar{f}_{kt}^2$ as prescribed above.

Case (I): $U_{ij} = U_{kt} \in O \subset U^1$. In this case, $y_i \notin B$ and $y_k \notin B$. Indeed, $y_i \in O_x$ and $y_k \in O_x$ for some x , $1 \leq x \leq n_1$. In this case, it follows from Property (6) of the properties of U^1 that for each $u \in G(y_i)$, $y_k \in \phi(u)$, and for each $v \in G(y_k)$,

$y_i \in \phi(v)$ since $O_x \supset \phi(U_{ij}) = \phi(U_{kt})$ and $\cap Q_x \supset O_x$ as defined above. Consequently, $G(y_i) = G(y_k)$ and $s_i = s_k$ contrary to the assumption above.

Case (II): $U_{ij} = U_{kt} \in H \subset U^1$, $y_i \in B$, and $y_k \in B$. Recall that $c_i = \min\{t \mid y_i \in \cap D_t\}$ and $s_i = \min\{s \mid \text{int } K_s \supset \overline{\phi(u)} \text{ for all } u \in U_1^1 \text{ such that } u \in G(y_i)\}$. Also, $c_k = \min\{t \mid y_k \in \cap D_t\}$ and $s_k = \min\{s \mid \text{int } K_s \supset \overline{\phi(u)} \text{ for all } u \in U_1^1 \text{ such that } u \in G(y_k)\}$. Furthermore, $U_{kt} \in H$ where $\phi(U_{kt}) \cap (\cap D_{c_k}) \neq \emptyset$ and $U_{kt} = U_{ij} \in H$ where $\phi(U_{kt}) \cap (\cap D_{c_i}) \neq \emptyset$. Now, $y_i \in \phi(U_{kt}) = \phi(U_{ij})$ and $y_k \in \phi(U_{ij})$.

Clearly, $y_i \in \phi(u)$ for each $u \in G(y_i)$. Since $y_i \in \phi(U_{ij})$, it follows by construction of U^1 that $\phi(h) \supset \phi(U_{ij})$ for all $h \in U_1^1$ such that there is no $B_x \in D_{c_i}$ such that $\partial\phi(h) \supset B_x$ and $\phi(h) \cap \phi(U_{ij}) \neq \emptyset$. If there is $B_x \in D_{c_i}$ such that $\partial\phi(u) \supset B_x$, then $y_i \in B_x$ and $y_i \notin \phi(u)$. Consequently, u is such an h and $\phi(u) \supset \phi(U_{ij}) = \phi(U_{kt})$ and $y_k \in \phi(u)$. Now, $y_k \in \phi(v)$ for each $v \in G(y_k)$. It follows in a similar way that $y_i \in \phi(v)$. Thus, $G(y_i) = G(y_k)$ and $s_i = s_k$ contrary to the assumption above.

It should be clear that α_1 and β_1 are well defined. Since V^2 covers M and refines U^1 which is irreducible, β_1 is defined for each $u \in U^1$. Observe that π_1 maps distinguished families onto distinguished families.

The first steps in the proof of Lemma 4 are complete.

With V^i defined (as indicated for $i = 1$ and 2), define U^i in the manner that U^1 is defined for V^1 . Use Lemma 3 to obtain a finite open covering R^{i+1} of Y satisfying the conditions of Lemma 3 where V^i replaces W_2 and R^{i+1} replaces R such that R^{i+1} generates a special covering V^{i+1} of M having the properties similar to those described above for V^2 w.r.t. U^1 and V^1 but w.r.t. U^i and V^i . Similarly, define α_i , β_i , and π_i in the manner that α_1 , β_1 , and π_1 are described above. Extend α_i , β_i , and π_i in the usual manner to the nerves $N(V^{i+1})$, $N(U^i)$, and $N(V^i)$, respectively.

It should be clear that the proof of Lemma 4 can be completed using mathematical induction and the methods employed above.

Orientation Of The Simplices In $N(V^m)$

Next, orient the simplices in $N(V^m)$ for each special covering V^m . Recall that for each special covering V^m , there is associated a covering R^m of Y which generates V^m . Let $R = R^m$ and $V = V^m$. Suppose that $(v_0, v_1, v_2, \dots, v_k) = \sigma^k$ is a k -simplex in $N(R)$. For each i , $0 \leq i \leq k$, $\phi^{-1}(v_i) = v_{i1} \cup v_{i2} \cup \dots \cup v_{it_i}$ where $\{v_{i1}, v_{i2}, \dots, v_{it_i}\}$ is the distinguished family determined by v_i . Consequently, σ^k determines a distinguished family of k -simplices in $N(\phi^{-1}(R))$ where $\phi^{-1}(R) = \{c \mid c \text{ is a component of } \phi^{-1}(r) \text{ for } r \in R\}$. If $q \in N[\sigma^k]$, the nucleus or carrier of σ^k , then $\phi^{-1}(q) \cap v_{ij} \neq \emptyset$ for each $i = 1, 2, \dots, k$ and each $j = 1, 2, \dots, t_i$. The orientation of a k -simplex $\delta^k = (v_{0j_0}, v_{1j_1}, \dots, v_{kj_k})$ is to be that of σ^k as indicated by the given order of the vertices $(v_{0j_0}, v_{1j_1}, \dots, v_{kj_k})$ of δ^k . Since $\phi^{-1}(v_i)$ has t_i components, there will be at least t_i k -simplices in $N(\phi^{-1}(R))$ which are mapped to σ^k by the simplicial mapping $\phi^* : N(\phi^{-1}(R)) \rightarrow N(R)$ induced by ϕ . This collection of k -simplices is the *distinguished family determined by σ^k* (more precisely, determined by $N[\sigma^k]$). The distinguished family of k simplices in $N(V^m)$ has cardinality p^c for some natural number c . Note that $N[\sigma^k]$ is connected and ulc.

Distinguished Families Of n -Simplices In $N(V^m)$

The distinguished families $F_i^1 = \{f_{ij}^1\}_{j=1}^{t_i^1}$ of members of the covering V^1 generate distinguished families of n -simplices. That is, for distinguished families $F_{k_i}^1$, $0 \leq i \leq n$, in V^1 such that $\bigcap_{i=1}^n (F_{k_i}^1)^* \neq \emptyset$ where $(F_{k_i}^1)^* = \bigcup_{j=1}^{t_i^1} f_{k_i j}^1$, the $F_{k_i}^1$, $0 \leq i \leq n$, generate a distinguished family of n -simplices consisting of all n -simplices $\{f_{k_0 j_0}^1, f_{k_1 j_1}^1, \dots, f_{k_n j_n}^1\}$ such that (1) $f_{k_i j_i}^1 \in F_{k_i}^1$, $0 \leq i \leq n$, and (2) $\bigcap_{i=0}^n f_{k_i j_i}^1 \neq \emptyset$. The number of n -simplices in this family is $\max\{t_{k_i}^1 \mid 0 \leq i \leq n\}$. As pointed out above, a distinguished family of n -simplices is determined by an n -simplex σ^n in $N(R)$ and such a family is a lifting of σ^n to $N(V^m)$.

Consider a distinguished family S_k^1 of n -simplices in $N(V^1)$ as defined above such that a distinguished family S_q^2 of n -simplices defined similarly in $N(V^2)$ using distinguished families in V^2 is mapped onto S_k^1 by $\pi_1^* : N(V^2) \rightarrow N(V^1)$. By construction, each n -

simplex in S_k^1 is the image of exactly p^c n -simplices for fixed c , a non negative integer, where $\text{card } S_q^2 = p^{c_q}$, $\text{card } S_k^1 = p^{c_k}$, and $c = c_k - c_q$. Of course, as a chain in $N(V^1)$, this is a trivial n -chain using coefficients in Z_p .

For each natural number m , define distinguished families S_i^m of n -simplices in $N(V^m)$ as described above. Each such family S_i^m is the lifting of an n -simplex γ_i^n in $N(R^m)$, that is, $\phi : M \rightarrow M/A_p$ induces a mapping $\phi^* : N(V^m) \rightarrow N(R^m)$. If γ_i^n is an n -simplex in $N(R^m)$, then $(\phi^*)^{-1}(\gamma_i^n)$ is the union of a distinguished family of n -simplices S_i^m in $N(V^m)$. The family S_i^m is the lifting of γ_i^n in $N(V^m)$. Each member of S_i^m has the same orientation as γ_i^n .

Observe that if S_i^m and S_j^m are two non degenerate distinguished families of n -simplices such that $S_i^m = \{\delta_{it}^n\}_{t=1}^{p^{c_i}}$, $S_j^m = \{\delta_{jt}^n\}_{t=1}^{p^{c_j}}$, and δ_{it}^n shares an $(n-1)$ -face with δ_{js}^n for some j and s , then for each t , $1 \leq t \leq p^{c_i}$, there is some s , $1 \leq s \leq p^{c_j}$, such that δ_{it}^n shares an $(n-1)$ -face with δ_{js}^n and, conversely, for each s , $1 \leq s \leq p^{c_j}$, there is some t , $1 \leq t \leq p^{c_i}$, such that δ_{js}^n and δ_{it}^n share an $(n-1)$ -face. If $c_i < c_j$, then for each t , $1 \leq t \leq p^{c_i}$, δ_{it}^n shares an $(n-1)$ -face with exactly $p^{c_j - c_i}$ members of δ_j^m .

The n -Skeleton, $N(V_n^m)$, Of $N(V^m)$, And Inverse Limit $H_n(V_n^m) \cong Z_p$

Let $N(V_n^m)$ denote the n -skeleton of $N(V^m)$. Recall that V^m denotes the special m^{th} covering. The special projections $\pi_m^* : N(V^{m+1}) \rightarrow N(V^m)$, factors by $\alpha_m^* : N(V^{m+1}) \rightarrow N(U^m)$ and $\beta_m^* : N(U^m) \rightarrow N(V^m)$ where $\pi_m^* = \beta_m^* \alpha_m^*$.

Let $H_n(V_n^m)$ denote the n^{th} simplicial homology of $N(V_n^m)$, the n -skeleton of $N(V^m)$. The coefficient group is always Z_p . The following two lemmas give facts concerning the homology which will be needed to finish the proof of the Theorem.

First, consider the following commutative diagram:

$$\begin{array}{ccccccc}
\leftarrow & H_n(V^m) & \xleftarrow{\beta_m^*} & H_n(U^m) & \xleftarrow{\alpha_m^*} & H_n(V^{m+1}) & \xleftarrow{\beta_{m+1}^*} & H_n(U^{m+1}) & \leftarrow \\
& \nu_m \uparrow & \beta_m^* \swarrow & & \nwarrow \alpha_m^* & \nu_{m+1} \uparrow & \beta_{m+1}^* \swarrow & & \nwarrow \alpha_{m+1}^* \\
\leftarrow & H_n(V_n^m) & \xleftarrow{\pi_m^*} & & H_n(V_n^{m+1}) & \xleftarrow{\pi_{m+1}^*} & & H_n(V_n^{m+2}) & \leftarrow
\end{array}$$

Here ν_m is the natural map of a cycle in $H_n(V_n^m)$ into its homology class in $H_n(V^m)$. The other maps are those induced by the projections α_m , β_m and π_m . The upper sequence, of course, yields the Čech homology group $\check{H}_n(M)$ as its inverse limit. Furthermore, it can be easily shown, using the diagram, that $\check{H}_n(M)$ is isomorphic to the inverse limit $G = \varprojlim H_n(V_n^m)$, of the lower sequence. Specifically, $\gamma : \check{H}_n(M) \rightarrow G$ defined by $\gamma(\Delta) = \{\beta_m^*(\pi_{U^m}(\Delta))\}$ (where Δ is a generator of $\check{H}_n(M)$) is an isomorphism of $\check{H}_n(M)$ onto G . We shall use the isomorphism in what follows and for convenience we shall let $\gamma(\Delta) = \{z_m^n(\Delta)\}$, i.e. $z_m^n(\Delta) = \beta_m^*(\pi_{U^m}(\Delta)) \in H_n(V_n^m)$.

Let Δ be the generator of $\check{H}(M)$ where $\Delta = \{z_m^n(\Delta)\}$, a sequence of n -cycles such that $\pi_m^* : z_{m+1}^n(\Delta) \rightarrow z_m^n(\Delta)$ where $z_m^n(\Delta)$ is the m^{th} coordinate of Δ , i.e., $\pi_{V_n^m}(\Delta) = z_m^n(\Delta)$ an n -cycle in $N(V_n^m)$ where $N(V_n^m)$ is the n -skeleton of $N(V^m)$. If $s\Delta$, $s \in \mathbb{Z}_p$, is any n -cycle in $\check{H}(M)$, then the coordinate n -cycles $z_m^n(s\Delta)$ and $z_m^n(\Delta)$ contain exactly the same n -simplices in $N(V_n^m)$ (see Lemma 7 below).

It follows from Lemma 2 that there is no loss of generality in assuming that $\pi_{V_n^m} : \check{H}(M) \rightarrow H_n(V_n^m)$ has the property that $\pi_{V_n^m}(\check{H}(M)) \cong \mathbb{Z}_p$ for each natural number m .

A Cernavskii Operator σ

4. *An Operator σ is defined on n -Chains similar to the Cernavskii operator in [27].*

Choose $g \in A_p - H_1$. Next, define an operator σ [cf. 27, 10] on n -chains. If δ^n is an n -simplex in $N(V^{m+1})$, then δ^n is in a unique non degenerate distinguished family S_i^{m+1} of n -simplices which has cardinality p^{c_i} . If $\pi_m^*(\delta^n)$ is an n -simplex, then each n -simplex in S_i^{m+1} projects by π_m^* to an n -simplex in $N(V^m)$. If $\pi_m^*(\delta^n)$ is a k -simplex, $k < n$, then each n -simplex in S_i^{m+1} projects under π_m^* to a k -simplex. Let $\sigma(\delta^n) = \sum_{s=0}^{p-1} g^s(\delta^n)$ where g^0 is the identity homeomorphism. Observe that if δ^n is an n -simplex in $N(V^{m+1})$ and $\pi_m^*(\delta^n)$ is an n -simplex in $N(V^m)$, then $\pi_m^*(\delta^n)$ is in a non degenerate distin-

guished family S_j^m of n -simplices in $N(V^m)$. Also, π_m^* maps $\{g^s(\delta^n)\}_{s=0}^{p-1}$ one-to-one onto $\{\pi_m^*(g^s(\delta^n))\}_{s=0}^{p-1} = \{g^s(\pi_m^*(\delta^n))\}_{s=0}^{p-1}$ and $\pi_m^*\sigma_m\delta^n = \sigma_m\pi_m^*\delta^n$. It should be clear that if δ^n is an n -simplex in $N(V^{m+1})$, then $\pi_m^*\sigma_m\delta^n = \pi_m^*\sum_{s=0}^{p-1}g^s(\delta^n) = \sum_{s=0}^{p-1}\pi_m^*g^s(\delta^n) = \sum_{s=0}^{p-1}g^s(\pi_m^*(\delta^n)) = \sigma_m\pi_m^*\delta^n$. Clearly, σ_m can be extended to any n -chain in $N(V_n^m)$. If δ^n is an n -simplex in $N(V_n^m)$ and $\pi_m^*\delta^n$ is a k -simplex, $k < n$, then $\pi_m^*\sigma_m(\delta^n)$ is a trivial n -chain.

Recall that the members of a distinguished family of k -simplices in $N(V^m)$ have the same orientation (being the lifting of a k -simplex in $N(R^m)$ where R^m is a certain cover of Y).

Lemma 5. [cf. 27] The special operator σ_m maps n -cycles to n -cycles and σ_m commutes with the special projections on n -chains. If $z_{m+1}^n(\Delta) = z$ is a coordinate n -cycle in $N(V_n^{m+1})$, then either (a) $\sigma_m z = 0$ or (b) $\sigma(z) = z$.

Proof. The homeomorphism g induces a simplicial homeomorphism \bar{g} of $N(V^{m+1})$ onto itself since g maps “vertices” (elements of V^{m+1}) one-to-one onto “vertices”.

Let Δ be a non zero n -cycle in $\check{H}_n(M) = \varprojlim H_n(V_n^{m+1}) \cong Z_p$. Let $z = z_{m+1}^n(\Delta)$ be the coordinate n -cycle of Δ in $H_n(V_n^{m+1})$. Let $\sigma = \sigma_m$. Consider $\sigma(z)$ where $z = \sum_{i=1}^k c_i \delta_i^n$. Thus, $\sigma z = \sum_{i=1}^k c_i \sigma \delta_i^n$ where $\sigma \delta_i^n = \sum_{s=0}^{p-1} g^s(\delta_i^n)$ and $\{g^s(\delta_i^n)\}_{s=0}^{p-1}$ is the distinguished subfamily of n -simplices in $N(V^{m+1})$ associated with δ_i^n . It follows by the construction that there is $z_{m+2}^n(\Delta)$, a coordinate n -cycle in $N(V_n^{m+2})$ such that $g^s(z_{m+2}^n(\Delta))$ maps by π_{m+1}^* onto $g^s(z)$. Hence, $g^s(z)$ for $0 \leq s \leq p-1$ contains the same n -simplices as z (see Lemma 7). Hence, $\sigma z = \sum_{t=0}^{p-1} z_t$ where $z_0 = z$ and $z_t = \sum_{i=1}^k c_i g^t(\delta_i^n)$. Since g is a homeomorphism on M , g induces an automorphism on $\pi_{V^{m+1}}(G)$ in $H_n(V_n^m)$ where $G = \varprojlim H_n(V_n^m) \cong Z_p$, $\pi_{V_n^{m+1}}(G) \cong Z_p$, and z_t is an n -cycle. It follows that $\sigma z = \sum_{s=0}^{p-1} g^s(z)$ which is an n -cycle. If g induces the identity automorphism, then $\sigma z = pz = 0 \pmod p$ (the trivial n -cycle). If the induced automorphism is not the

identity, then it will be shown that $\sum_{t=1}^{p-1} z_t = 0 \pmod p$ and that $\sigma z = z$, that is, σ is the identity automorphism. Now, $\pi_{V_n^{m+1}}(G) \cong Z_p = \{0, 1, 2, \dots, p-1\}$. Since g induces an automorphism on $\pi_{V_n^{m+1}}(G)$, it induces an automorphism g_* on Z_p . Let $g_*(1) = x$. Hence, $g_*^s(1) = x^s$. It is well known in number theory that p divides $x^{p-1} - 1$. Also, $x^{p-1} - 1 = (x-1)(1+x+x^2+\dots+x^{p-2})$. Consequently, p divides $x(1+x+x^2+\dots+x^{p-2}) = x+x^2+\dots+x^{p-1} = \sum_{s=1}^{p-1} x^s = 0 \pmod p$. It follows that $\sigma(z) = \sum_{s=0}^{p-1} g^s(z) = z + \sum_{s=1}^{p-1} g^s(z) = z \pmod p$. Thus, if g does not induce the identity automorphism, then σ is the identity automorphism.

If under the projection $\pi_m^* : H_n(V_n^{m+1}) \rightarrow H_n(V_n^m)$, the image of an n -simplex δ^n is a k -simplex with $k < n$, then the same is true for all members of the distinguished family to which δ^n belongs. Thus, $\pi_m^* \sigma_m \delta^n = \sigma_m \pi_m^* \delta^n = 0$ where $\sigma_m \pi_m^* \delta^n$ is defined above and is a trivial n -chain. If $\pi_m^*(\delta^n)$ is an n -simplex, then by construction the distinguished subfamily of n -simplices with which δ^n is associated is in one-to-one correspondence with the distinguished subfamily of n -simplices with which $\pi_m^*(\delta^n)$ is associated. Thus, $\pi_m^* \sigma_m \delta^n = \sigma_m \pi_m^* \delta^n$. Consequently, σ carries over to the n -cycles of M and to $\check{H}(M)$. Lemma 5 is proved.

Lemma 6. If Δ_1 and Δ_2 are non zero elements of $\check{H}_n(M)$, then for each m , exactly the same simplices appear in the chains $z_m^n(\Delta_1)$ and $z_m^n(\Delta_2)$ which are in the n -dimensional complex, $N(V_n^m)$, the n -skeleton of $N(V^m)$.

Proof. Suppose that this is not true. Without loss of generality, assume that the n -simplex δ^n appears in $z_m^n(\Delta_1)$ and not in $z_m^n(\Delta_2)$ where $z_m^n(\Delta_1)$ and $z_m^n(\Delta_2)$ are the m^{th} coordinates of Δ_1 and Δ_2 , respectively. Since $N(V_n^m)$ is n -dimensional, these coordinates are n -cycles in $N(V_n^m)$. Since $\check{H}_n(M) = Z_p$, assume that Δ_1 generates $\check{H}_n(M)$ and that $\Delta_2 = s\Delta_1$ for some natural number s , $1 \leq s < p$. Consequently, $z_m^n(\Delta_2) = s(z_m^n(\Delta_1))$. It follows that δ^n appears in $s(z_m^n(\Delta_1))$ and hence in $z_m^n(\Delta_2)$ – a contradiction.

Lemma 7. Suppose that $\Delta \in \check{H}_n(M)$ with $\Delta \neq 0$ and $z = z_{m+1}^n(\Delta) = \pi_{V_n^{m+1}}(\Delta)$, the

coordinate n -cycle of Δ in $H_n(V_n^{m+1})$. Let $z = \sum_{i=1}^q c_i \delta_i^n$. Then the collection $C_j = \{\delta_{j_1}^n, \delta_{j_2}^n, \dots, \delta_{j_{t_j}}^n\}$ of all n -simplices in $\{\delta_1^n, \delta_2^n, \dots, \delta_q^n\}$ which are in a fixed distinguished family S_j^{m+1} of n -simplices in $N(V_n^{m+1})$ have the properties (1) if $x = \sum_{i=1}^t c_{j_i} \delta_{j_i}^n$, then either (a) $\sigma_m x = 0$ when g induces the identity automorphism or (b) $\sigma_m x = x$ when g does not induce the identity automorphism and (2) $\sum_{i=1}^t c_{j_i} = 0 \pmod p$.

Proof. By Lemma 5, either (a) $\sigma_m z = 0$ or (b) $\sigma_m(z) = z$.

Case (a): Since $\sigma_m(z) = 0$, it follows that $\sigma_m x = 0$ since for each i , $1 \leq i \leq t$, $\sigma_m \delta_{j_i}^n = \sum_{s=0}^{p-1} g^s(\delta_{j_i}^n)$ where for each s , $0 \leq s \leq p-1$, $g^s(\delta_{j_i}^n)$ is an n -simplex in $C_q \subset S_j^{m+1}$. Choose notation such that (1) $S_i^{m+1} = \{\delta_j\}_{j=1}^{p^k}$ ($p^k = p^{c_j}$ in earlier notation), (2) $g\delta_j = \delta_{j+1}$ for $1 \leq j < p^k$ and $g\delta_{p^k} = \delta_1$ (g permutes the δ_j in a cyclic order), and (3) $x = \sum_{i=1}^{p^k} c_i \delta_i$ where $c_i = 0$ iff $\delta_i \notin C_q$ and $c_i = c_{j_i}$ iff $\delta_i = \delta_{j_i}^n \in C_q$. Let $\sigma = \sigma_m$. Since $\sigma(z) = 0$, it follows that $\sigma(x) = 0$. Note that $0 = \sigma(x) = c_1 \sum_{i=1}^p \delta_i + c_2 \sum_{i=1}^p \delta_{i+1} + \dots + c_j \sum_{i=1}^p \delta_{i+j-1} + \dots + c_{p^k-p} \sum_{i=1}^p \delta_{i+p^k-p-1} + \dots + c_{p^k} (\delta_{p^k} + \delta_1 + \delta_2 + \dots + \delta_{p-1})$. Rearrange as

$$\begin{aligned} \sigma(x) &= (c_1 + c_{p^k} + c_{p^k-1} + \dots + c_{p^k-p+2})\delta_1 + \\ &\quad (c_2 + c_1 + c_{p^k} + \dots + c_{p^k-p+1})\delta_2 + \\ &\quad \vdots \\ &\quad (c_p + c_{p-1} + c_{p-2} + \dots + c_1)\delta_p + \\ &\quad (c_{p+1} + c_p + c_{p-1} + \dots + c_2)\delta_{p+1} + \\ &\quad \vdots \\ &\quad (c_{p^k} + c_{p^k-1} + \dots + c_{p^k-p+1})\delta_{p^k}. \end{aligned}$$

Since $\sigma(x) = 0$, it follows that the coefficient of δ_i is $0 \pmod p$ for $1 \leq i \leq p^k$. A careful consideration of pairs of successive coefficients of δ_i and δ_{i+1} will give the following result. If $1 \leq i \leq p^k$, $1 \leq j \leq p^k$, and $i \equiv j \pmod p$, then $c_i \equiv c_j \pmod p$. Thus

$\sum_{i=1}^{p^k} c_i = \sum_{i=1}^p c_i + \sum_{i=p}^{2p} c_i + \sum_{i=2p+1}^{3p} c_i + \cdots + \sum_{i=(p^{k-1}-1)p}^{(p^{k-1})p} c_i$ with p^{k-1} summations each of length p . Now, $c_i \equiv c_j \pmod p$ if $i \equiv j \pmod p$ gives that each of the p^{k-1} summations is congruent to 0 mod p . Thus, $\sum_{i=1}^{p^k} c_i = p^{k-1} \left(\sum_{i=1}^p c_i \right) \equiv 0 \pmod p$ if $k > 1$. If $k = 1$, then

$$x = \sum_{i=1}^p c_i \delta_i,$$

$$\begin{aligned}
 \sigma(x) &= c_1(\delta_1 + \delta_2 + \cdots + \delta_p) \\
 &\quad + c_2(\delta_2 + \delta_3 + \cdots + \delta_{p+1}) \\
 &\quad \vdots \\
 &\quad + c_p(\delta_p + \delta_1 + \delta_2 + \cdots + \delta_{p-1}), \text{ rearrange as} \\
 &(c_1 + c_2 + \cdots + c_p)\delta_1 + \\
 &(c_1 + c_2 + \cdots + c_p)\delta_2 + \\
 &\quad \vdots \\
 &(c_1 + c_2 + \cdots + c_p)\delta_p,
 \end{aligned}$$

and $\sum_{i=1}^p c_i \equiv 0 \pmod p$.

It is instructive to consider a simple example. Let $p = 3$ and $x = \sum_{i=1}^9 c_i \delta_i$. Thus,

$$\begin{aligned}
 \sigma x &= c_1(\delta_1 + \delta_2 + \delta_3) + c_2(\delta_2 + \delta_3 + \delta_4) + c_3(\delta_3 + \delta_4 + \delta_5) + c_4(\delta_4 + \delta_5 + \delta_6) + c_5(\delta_5 + \delta_6 + \delta_7) \\
 &\quad + c_6(\delta_6 + \delta_7 + \delta_8) + c_7(\delta_7 + \delta_8 + \delta_9) + c_8(c_8 + c_9 + c_1) + c_9(c_9 + \delta_1 + \delta_2) = \text{(by rearrangement)} \\
 &= (c_1 + c_8 + c_9)\delta_1 + (c_1 + c_2 + c_9)\delta_2 + (c_1 + c_2 + c_3)\delta_3 + (c_2 + c_3 + c_4)\delta_4 + \\
 &\quad (c_3 + c_4 + c_5)\delta_5 + (c_4 + c_5 + c_6)\delta_6 + (c_5 + c_6 + c_7)\delta_7 + (c_6 + c_7 + c_8)\delta_8 + (c_7 + c_8 + c_9)\delta_9.
 \end{aligned}$$
 For each i , $1 \leq i \leq 9$, the coefficient of $\delta_i \equiv 0 \pmod 3$. Observe that from the coefficients of δ_1 and δ_2 , it follows that $c_2 \equiv c_8 \pmod 9$. The coefficients of δ_2 and δ_3 yield that $c_3 \equiv c_9 \pmod 3$. Continuing, $c_1 \equiv c_4$, $c_2 \equiv c_5$, $c_3 \equiv c_6$, $c_4 \equiv c_7$, $c_5 \equiv c_8$, $c_6 \equiv c_9$, and $c_7 \equiv c_1 \pmod 3$. Thus, $(c_1 + c_2 + c_3) + (c_4 + c_5 + c_6) + (c_7 + c_8 + c_9) \equiv 0 \pmod 3$ since $(c_4 + c_5 + c_6) \equiv (c_1 + c_2 + c_3) \pmod 3$, $(c_7 + c_8 + c_9) \equiv (c_4 + c_5 + c_6) \equiv (c_1 + c_2 + c_3) \pmod 3$.

3 and $\sum_{i=1}^9 c_i \equiv 3(c_1 + c_2 + c_3) \equiv 0 \pmod{3}$.

Case (b): $\sigma_m(x) = x$. Choose notation as in Case (a). Write $\sigma(x)$ as in Case (a), but in this case, $x = \sigma(x)$ rather than $0 = \sigma(x)$. Consider first the example $p = 3$ and $x = \sum_{i=1}^9 c_i \delta_i$ where $k = 2$. Now, $\sigma x = (c_1 + c_8 + c_9)\delta_1 + (c_1 + c_2 + c_9)\delta_2 + (c_1 + c_2 + c_3)\delta_3 + (c_2 + c_3 + c_4)\delta_4 + (c_3 + c_4 + c_5)\delta_5 + (c_4 + c_5 + c_6)\delta_6 + (c_5 + c_6 + c_7)\delta_7 + (c_6 + c_7 + c_8)\delta_8 + (c_7 + c_8 + c_9)\delta_9 = x = c_1\delta_1 + c_2\delta_2 + c_3\delta_3 + c_4\delta_4 + c_5\delta_5 + c_6\delta_6 + c_7\delta_7 + c_8\delta_8 + c_9\delta_9$. This is an identity. Thus, the coefficient of δ_i on one side is equal mod p to the coefficient of δ_i on the other side. Hence, $c_1 + c_8 + c_9 \equiv c_1 \pmod{p}$, $c_1 + c_2 + c_9 \equiv c_2 \pmod{p}$, $c_1 + c_2 + c_3 \equiv c_3 \pmod{p}$, and so forth. Thus, $\sum_{i=1}^9 c_i \equiv (c_1 + c_8 + c_9) + (c_1 + c_2 + c_9) + (c_1 + c_2 + c_3) + (c_2 + c_3 + c_4) + (c_3 + c_4 + c_5) + (c_4 + c_5 + c_6) + (c_5 + c_6 + c_7) + (c_6 + c_7 + c_8) + (c_7 + c_8 + c_9) = 3(c_1 + c_2 + c_3 + c_4 + c_5 + c_6 + c_7 + c_9) \equiv 0 \pmod{p}$.

Consider the general case as in Case (a) but with $x = \sigma(x)$ rather than $0 = \sigma(x)$. Hence,

$$\begin{aligned} x = \sigma(x) &= (c_1 + c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+2})\delta_1 + \\ &\quad (c_2 + c_1 + c_{p^k} + \cdots + c_{p^k-p+1})\delta_2 + \\ &\quad \vdots \\ &\quad (c_p + c_{p-1} + c_{p-2} + \cdots + c_1)\delta_p + \\ &\quad (c_{p+1} + c_p + c_{p-1} + \cdots + c_2)\delta_{p+1} + \\ &\quad \vdots \\ &\quad (c_{p^k} + c_{p^k-1} + \cdots + c_{p^k-p+1})\delta_{p^k} = \sum_{i=1}^{p^k} c_i \delta_i. \end{aligned}$$

It follows from this identity that the coefficient of δ_i on one side is equal mod p to the coefficient of δ_i on the other side. Consequently, $\sum_{i=1}^{p^k} c_i = p \sum_{i=1}^{p^k} c_i \pmod{p} = 0 \pmod{p}$ as claimed where $k > 1$. For $k = 1$, $x = \sum_{i=1}^p c_i \delta_i = \left(\sum_{i=1}^p c_i \right) \delta_1 + \left(\sum_{i=1}^p c_i \right) \delta_2 + \cdots +$

$\left(\sum_{i=1}^p c_i\right) \delta_p$, $c_t \equiv \sum_{i=1}^p c_i \pmod p$ for each t , $1 \leq t \leq p$, and $\sum_{i=1}^p c_i \equiv p \left(\sum_{i=1}^p c_i\right) \pmod p = 0$. Lemma 7 is proved.

5. *A proof that a p -adic group A_p can not act freely on a compact connected n -manifold where $\phi : M \rightarrow M/A_p$ is the orbit mapping.*

Remarks. If the compact connected n -manifold M has a non empty boundary, then two copies of M can be sewed together by identifying the boundaries in such a way that the result is a compact connected n -manifold M' without boundary. If A_p acts effectively on M , then A_p acts effectively on M' . There is no loss of generality in assuming that M is a compact connected n -manifold without boundary.

Definition. An n -manifold (M, d) is said to have Newman's Property w.r.t. the class $L(M, p)$ (as stated above) iff there is $\epsilon > 0$ such that for any $\phi \in L(M; p)$, there is some $x \in M$ such that $\text{diam } \phi^{-1}\phi(x) \geq \epsilon$.

Generalizations can be made to metric spaces (X, d) which are locally compact, connected, and lc^n [4] which have domains D such that \bar{D} is compact, lc^n , and $H_n(X, X - D), Z_p) \cong Z_p$.

Theorem. If $L(M, p)$ is the class of all orbit mappings $\phi : M \rightarrow M/A_p$ where A_p acts freely on a compact connected n -manifold M , then M has Newman's Property w.r.t. $L(M, p)$.

Proof. There is no loss of generality in assuming that M has empty boundary.

By hypothesis, $\check{H}_n(M) \cong Z_p$. Consider a finite open covering $U = W_1$ where W_1 and W_2 satisfy Lemma 2 and such that if $z(\Delta)$ is the V -coordinate of a non-zero n -cycle $\Delta \in \check{H}_n(M)$ where V refines W_2 , then $\pi_{VU} z(\Delta) \neq 0$. Let ϵ be the Lebesgue number of W_2 . Suppose that there is $\phi \in L(M, p)$ such that $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. Construct the special coverings $\{V^m\}$ and the special refinements $\{U^m\}$ as in Lemma 4 such that the star of each distinguished family of V^1 lies in some element of

W_2 . Furthermore, the special projections π_m can be constructed such that if $\{\delta_{sj}^n\}_{j=1}^{t_s^n}$ is a distinguished family of n -simplices in $N(V_n^m)$, then $\pi_{V^m U}$ takes δ_{sj}^n , $1 \leq j \leq t_s$, to the same simplex δ_s in $N(U)$. Now, let $z_m = z_m^n(\Delta)$.

Let $z_m = \sum_{i=1}^k c_i \delta_i^n$. Hence, for each j , $1 \leq j \leq k$, δ_j^n is in a non degenerate distinguished family S_j^m of n -simplices in $N(V^m)$. Let $C_j = \{\delta_{j_1}^n, \delta_{j_2}^n, \dots, \delta_{j_t}^n\}$ denote the collection of all n -simplices such that (1) $\delta_{j_i}^n$ appears in z_m for $1 \leq i \leq t$ and (2) $S_j^m \supset C_j$. By Lemma 7, $\sum_{i=1}^t c_{j_i} = 0 \pmod p$. Since the n -simplices in C_j are sent by $\pi_{V^m U}$ to a single simplex δ_j in $N(U)$, it follows that the coefficient of δ_j is $0 \pmod p$ and z is sent by $\pi_{V^m U}$ to the zero n -cycle in $N(U)$. Thus, the projection of z_m by $\pi_{V^m U} : H_n(V_n^m) \rightarrow H_n(U)$ takes the nontrivial n -cycle $z_m(\Delta)$ to the 0 n -cycle mod p . This violates the conclusion of Lemma 2. Thus, M has Newman's Property w.r.t. the class $L(M, p)$. Hence, ϵ is a Newman's number and the Theorem is proved.

It is well known that if A_p acts effectively on a compact connected n -manifold M , then given any $\epsilon > 0$, there is an effective action of A_p on M such that $\text{diam } \phi^{-1}\phi(x) < \epsilon$ for each $x \in M$. That is, M fails to have Newman's property w.r.t. $L(M, p)$. It follows that A_p can not act effectively on a compact connected n -manifold M .

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The State University of New York at Binghamton